A NOTE ON A MONTE-CARLO ESTIMATOR OF DARLING AND ROBBINS

BY

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ABSTRACT

Darling and Robbins, in discussing certain sequential tests that do not necessarily terminate with probability one, give a family of Monte-Carlo procedures for estimating the probability of termination. The choice of estimator is left open, although one presumably would like to have a small variance (the estimators are unbiased). As a contribution to this problem, we show that these estimators do not have moments higher than the first.

1. **Introduction.** Darling and Robbins [1] discuss certain sequential tests that do not necessarily terminate with probability one. Specifically, as a test for whether a normal mean, μ, is positive or not, they propose the following: Let $X_1, X_2 \cdots$ be iid $N(\mu, 1)$ random variables. Let $S_n = X_1 + \cdots + X_n$ and N be the first $n \geqq m$ so that $S_n \geqq a_n$, or be $+\infty$ if no such n occurs. Here $a_n = (2\lambda n \log\log n)^{\frac{1}{2}}$; $\lambda > 1$. One concludes the mean is positive if for some $n \geqq m$, $S_n \geqq a_n$; i.e., if $N < \infty$. The strong law of large numbers implies that for $\mu > 0$, $P_\mu(N < \infty) = 1$ and Darling and Robbins [1] show that for $\mu = 0$,

$$\alpha = P_0(N < \infty) = O([\log\log m]^{1/2} / [\log m]^{\lambda - 1}).$$

A way of obtaining a Monte-Carlo estimate of α is proposed by Darling and Robbins [1]: Let f_μ denote the $N(\mu, 1)$ density. Then for $\mu > 0$,

$$\begin{aligned}\alpha &= P_0(N < \infty) = \sum_n \int_{(N=n)} \prod_1^n f_0(x_i)dx_i \\ &= \sum_n \int_{(N=n)} \left[\prod_1^n \frac{f_0(x_i)}{f_\mu(x_i)}\right] \prod_1^n f_\mu(x_i)dx_i \\ &= E_\mu \prod_1^N [f_0(X_i)/f_\mu(X_i)] \\ &= E_\mu \exp\{-\mu S_{\omega_N} + N\mu^2/2\}.\end{aligned}$$

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Letting M = first $n \geqq m$ so that $S_n \geqq a_n - n\mu$,

$$\alpha = E_0 \exp\{-\mu S_M - M\mu^2/2\} = E_0 L, \text{ say.} \tag{2}$$

One can make (say) k successive Monte-Carlo determinations of M and hence L, obtaining $L_1, \cdots, L_k$. Then $\bar{L}_k = \sum_1^k L_i/k$ is an unbiased estimator of α and is consistent as $k \to \infty$. The question arises as to the best choice of μ; say that which gives L as small a variance as possible. Of course $E_0 M = E_\mu N$ decreases as μ increases, so that one prefers to use a large value of μ, to keep the cost of sampling small. As a contribution to this problem, we show that for every $t > 1$, $E_0 L^t = +\infty$. An optimality criterion may still be based on the first moment (mean absolute deviation, e.g.), balancing expected sample size against expected error. We do not enter into such considerations here.

2. Demonstration. We proceed to bound EL^t from below. It will be seen that the only condition required on a_n is that it increases with n and that $a_n = o(n)$. In the sequel, E means E_0. We begin by noting that

$$EL^t > EL^t 1_{(M>m)}. \tag{3}$$

We obtain a lower bound for the last quantity by considering $E(L^t \mid M, S_{M-1})$. By the definition of M, $S_M \geqq a_M - M\mu$, while on $(M > m)$, $S_{M-1} < a_{-1} - (M-1)\mu \leqq a_M - (M-1)\mu$. Writing $S_M = S_{M-1} + X_M$, the preceding shows that $X_M \geqq a_M - M\mu - S_{M-1}$ and on $(M > m)$, $a_M - M\mu - S_{M-1} \geqq -\mu$. Let $c(\mu, t) = \inf_{u > -\mu} E[\exp\{-\mu t(X - u)\{ \mid X \geqq u]$. It is easily checked that the infimum occurs at $u = -\mu$ and that $c(\mu, t) > 0$. Then on $(M > m)$,

$$\begin{aligned} E(L^t \mid M, S_{M-1}) &= E[\exp\{t(-\mu S_M - \mu M^2/2)\} \mid M, S_{M-1}] \\ &= \exp\{t(-\mu a_M + \mu^2 M/2)\} E[\exp\{-\mu t(X_M + S_{M-1} + M\mu - a_M)\} \mid M, S_{M-1}] \\ &\geqq c(\mu, t) \exp\{t(-\mu a_M + \mu^2 M/2)\}, \end{aligned} \tag{4}$$

since given M and S_{M-1}, X_M is a $N(0,1)$ variable constrained to exceed $a_M - M\mu - S_{M-1}$ and the latter quantity is larger than $-\mu$ on $(M > m)$. Thus from (3) and (4), we see that

$$EL^t \geqq c(\mu, t) E \exp\{t(-\mu a_M + \mu^2 M/2)\} 1_{(M>m)}. \tag{5}$$

Choose $\delta > 0$. We may choose $n_\delta \geqq m$ so that $n > n_\delta \to a_n \leqq \delta\mu n/2$. Then

$$\begin{aligned} &\exp\{-\mu t a_M + \mu^2 t M/2\} 1_{(M>n_\delta)} \\ &\qquad \geqq \exp\{t\mu^2(1-\delta)M/2\} 1_{(M>n_\delta)}. \end{aligned} \tag{6}$$

Together, (5) and (6) show that

$$E\exp\{t\mu^2(1-\delta)M/2\} = +\infty \Rightarrow EL^t = +\infty. \tag{7}$$

We investigate Ee^{sM} for $s > 0$. We note first that $(S_k/k < -\mu,\ k = 1,\cdots,n) \Rightarrow (M > n)$.

Let $W(t)$ denote standard Brownian motion. Since $\{S_n/n\}$ has the same joint distribution as $\{W(1/n)\}$, we see that $P(M > n) \geqq P(W(t) < -\mu,\ 1/n \leqq t \leqq 1)$. Standard results for Brownian motion imply

$$\begin{aligned} P(W(t) \leqq -\mu, 1/n \leqq t \leqq 1 \,|\, W(1/n) = -w) &= P(W(t) \leqq w-\mu,\ 0 \leqq t \leqq 1 - 1/n) \\ &\geqq P(W(t) \leqq w-\mu,\ 0 \leqq t \leqq 1) \\ &= \begin{cases} 2\Phi(w-\mu) - 1 & \text{if } w > \mu \\ 0 & \text{if } w \leqq \mu, \end{cases} \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Since $W(1/n) \sim N(0, 1/n)$, we see that for $\varepsilon > 0$,

$$\begin{aligned} P(M > n) &\geqq \int_{\mu}^{\infty} \sqrt{\frac{n}{2\pi}}[2\Phi(\omega-\mu) - 1]e^{-n\omega^2/2}d\omega \\ &\geqq [2\Phi(\varepsilon) - 1]\int_{\mu+\varepsilon}^{\infty} \sqrt{\frac{n}{2\pi}} e^{-n\omega^2/2}d\omega \\ &= [2\Phi(\varepsilon) - 1][1 - \Phi(\sqrt{n}(\mu+\varepsilon))]. \end{aligned}$$

For some $b > 0$ and sufficiently large x, $1 - \Phi(x) \geqq b\exp\{-x^2/2\}/x$. Hence or sufficiently large n,

$$P(M > n) \geqq c\exp\{-n(\mu+\varepsilon)^2/2\}/\sqrt{n}, \tag{8}$$

where c depends on ε and μ but not n. From (8), we see that $Ee^{sM} \geqq \sum_{k>1} P(e^{sM} > k) = \sum P(M > (\log k)/s) \geqq \sum (k^{-(\mu+\varepsilon)^2/2s})(s/\log k)^{1/2} = +\infty$ if $(\mu+\varepsilon)^2/2s \leqq 1$. Since ε is arbitrary, $Ee^{sM} = +\infty$ if $s > \mu^2/2$. Referring to (7), we see that $EL^t = +\infty$ if $t\mu^2(1-\delta)/2 > \mu^2/2$ or if $t(1-\delta) > 1$. Since δ is arbitrary, $EL^t = +\infty$ if $t > 1$.

An objection to L as a Monte-Carlo estimator of $P_0(N < \infty)$ should be noted. Let

$$L_n = \exp\{-\mu S_n - n\mu^2/2\}.$$

It is well-known that L_n is martingale. Thus $L = L_M$ is just this martingale stopped by M. Although $EL_n = 1$ for all n, M has the property that $EL_M < 1$. However, if this martingale sequence is stopped by a bounded stopping time, T, the expectation is preserved: $EL_T = 1$. Since in practice, any Monte-Carlo simulation would generate only bounded stopping times, it is not clear whether one can effectively estimate $P_0(N < \infty)$ in this way.

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References

1. D. Darling and H. Robbins, *Some further remarks on inequalities for sample sums*, Proc. Nat. Acad. Sci **60** (1968), 1175–1182.

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